



J.S. Bendat.

The Hilbert Transform

**and Applications to
Correlation Measurements**



Dr. Bendat received his Ph.D. degree in Mathematics from the University of Southern California and M.S. degree in Physics from the California Institute of Technology. Since 1970, Dr. Bendat has been an independent mathematical consultant specializing in random data analysis theory, experimental measurement techniques, error criteria, and physical applications. He has conducted research for many engineering companies in the United States and Europe in aerospace, oceanographic and automotive fields. Prior affiliations were with Measurement Analysis Corporation, TRW Systems, and Northrop Corporation. Besides lecturing at short courses, Dr. Bendat is the author or co - author of numerous technical papers and of four widely used books (all published by Wiley - Interscience, New York).

THE HILBERT TRANSFORM AND APPLICATIONS TO CORRELATION MEASUREMENTS

by

*Julius S. Bendat**, Ph.D.

* Julius S. Bendat is an independent mathematical consultant with offices at 833 Moraga Drive,
Los Angeles, CA 90049.

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1. INTRODUCTION

During the past twenty-five years, considerable attention has been devoted to Fourier transforms and its applications as published in References [1–7]. This has been accompanied by development of needed hardware systems and software programs to carry out Fourier computations and display the results as in References [8–11]. Similar extensive studies and developments were not carried out for Hilbert transforms and its applications, although certain special properties were established as reported in References [4–7, 12, 13, 16]. This paper has been written to bring about greater understanding of Hilbert transform techniques and to promote its application to engineering problems, particularly for measurements such as peak correlation functions.

Unlike the Fourier transform $X(f)$ which moves the independent variable of a signal $x(t)$ from the time domain to the frequency domain, the Hilbert transform leaves the signal $x(t)$ in the same domain. The Hilbert transform $\tilde{x}(t)$ of a real-valued time signal $x(t)$ is another real-valued time signal, and the Hilbert transform $\tilde{X}(f)$ of a complex-valued frequency function $X(f)$ is another complex-valued frequency function.

This paper contains the following material. Section 2 reviews definitions, examples and basic properties of Fourier transforms for comparison with Section 3 where definitions, examples and basic properties are stated for Hilbert transforms. Section 4 develops appropriate Hilbert transform formulas for correlation functions of stationary random data. Important applications of these results are outlined in Section 5 for three types of nondispersive propagation problems. Sections 6 and 7 then derive statistical error analysis criteria for magnitude and location estimates of peak correlation functions, their Hilbert transforms and their envelopes. These formulas are essential to design experimental test programs and to evaluate such measured results. Section 8 discusses dispersive propagation problems.

Illustrations of Hilbert transform techniques for applications as discussed here are given in References [14, 15], where the Bruel & Kjaer 2 Channel FFT Analyzers, Type 2032 or 2034 are used. In these analyzers, the Hilbert transform is accomplished by changing real-valued time functions (such as correlation functions, impulse response functions, etc.) into complex-valued time functions where the imaginary part of the new complex-valued time function is the Hilbert transform of the real part. Then, these complex-valued time domain functions (called analytic signals) can be displayed in terms of their magnitude and phase versus time. This is similar to plots of the magnitude and phase versus frequency of complex-valued frequency domain functions. The magnitude of the analytic signal describes the envelope of the original real-valued time domain signal, and the phase leads to an “instantaneous frequency” for signals sweeping in frequency with time. Different Hilbert transform applications with these Bruel & Kjaer analyzers are discussed and displayed in Reference [16].

2. FOURIER TRANSFORM FORMULAS

For convenience, all formulas will be stated in integral form over theoretical limits. In practice, of course, with digital processing of data, such integrals would be replaced by finite summations representing estimates of the desired theoretical quantities. Such finite summations always exist.

2.1. DEFINITION OF FOURIER TRANSFORM, MAGNITUDE AND PHASE

In agreement with References [1, 2], the **Fourier transform** of a real-valued function $x(t)$ extending from $-\infty < t < \infty$ is a complex-valued function $X(f)$ defined by

$$X(f) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad (1)$$

Assuming that the complex-valued $X(f)$ exists for all f over $-\infty < f < \infty$, the **inverse Fourier transform** brings $X(f)$ back to $x(t)$ as defined by

$$x(t) = \mathcal{F}^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \quad (2)$$

From Equation (1), one can define

$$X(f) = X_R(f) - jX_I(f) \quad (3)$$

where the real part $X_R(f)$ and the imaginary part $X_I(f)$ are

$$X_R(f) = \int_{-\infty}^{\infty} x(t) \cos 2\pi ft dt \quad (4)$$

$$X_I(f) = \int_{-\infty}^{\infty} x(t) \sin 2\pi ft dt \quad (5)$$

Also one can use complex polar notation to define a **magnitude** $|X(f)|$ and **phase** $\phi_x(f)$ by

$$\begin{aligned} X(f) &= |X(f)| e^{-j\phi_x(f)} \\ &= |X(f)| \cos \phi_x(f) - j |X(f)| \sin \phi_x(f) \end{aligned} \quad (6)$$

Thus

$$X_R(f) = |X(f)| \cos \phi_x(f) \quad (7)$$

$$X_I(f) = |X(f)| \sin \phi_x(f)$$

$$|X(f)| = [X_R^2(f) + X_I^2(f)]^{1/2} \quad (8)$$

$$\phi_x(f) = \tan^{-1} [X_I(f) / X_R(f)] \quad (9)$$

Note that in place of Equation (3), one could have defined

$$X(f) = X_R(f) + jX_I(f) \quad (10)$$

This gives the same $X_R(f)$ as in Equation (4) but a negative $X_I(f)$ from Equation (5), namely,

$$X_I(f) = - \int_{-\infty}^{\infty} x(t) \sin 2\pi f t \, dt \quad (11)$$

Also, in place of Equation (6), one could have defined

$$\begin{aligned} X(f) &= |X(f)| e^{j\phi_x(f)} \\ &= |X(f)| \cos \phi_x(f) + j |X(f)| \sin \phi_x(f) \end{aligned} \quad (12)$$

This gives the negative result

$$\phi_x(f) = -\tan^{-1} [X_I(f) / X_R(f)] \quad (13)$$

using the $X_I(f)$ of Equation (5). To avoid confusion and negative signs in $X_I(f)$ and $\phi_x(f)$, this paper will use the definitions of Equations (3) to (9) in place of Equations (10) to (13).

Similar definitions and phase relations will be used in later cross-spectral density functions. By these definitions time delays and phase angles are positive when data propagates from point A to point B.

2.2. EXAMPLES OF FOURIER TRANSFORMS

The following Table 1 gives several $x(t)$ and its associated $X(f)$. Many other such results are available in the References.

$x(t)$	$X(f)$
$\cos 2\pi f_o t$	$\frac{1}{2} [\delta(f-f_o) + \delta(f+f_o)]$
$\sin 2\pi f_o t$	$\frac{1}{2j} [\delta(f-f_o) - \delta(f+f_o)]$
$\frac{\sin t}{t}$	$\begin{cases} \pi & (-1/2\pi) \leq f \leq (1/2\pi) \\ 0 & \text{otherwise} \end{cases}$
$\frac{1}{1+t^2}$	$\begin{cases} \pi e^{-2\pi f} & f > 0 \\ \pi e^{2\pi f} & f < 0 \end{cases}$
$e^{-c t } \cos 2\pi f_o t$	$\frac{c}{c^2 + 4\pi^2(f-f_o)^2} + \frac{c}{c^2 + 4\pi^2(f+f_o)^2}$

Table 1. Examples of Fourier Transforms

2.3. PROPERTIES OF FOURIER TRANSFORMS

A number of properties will now be stated for Fourier transforms which follow readily from the basic definitions or are proved in References [1-7]. Let $X(f) = \mathcal{F}[x(t)]$ and $Y(f) = \mathcal{F}[y(t)]$ be Fourier transforms of $x(t)$ and $y(t)$.

a) Linear Property

$$\mathcal{F}[a x(t) + b y(t)] = a X(f) + b Y(f) \quad (14)$$

for any functions $x(t)$, $y(t)$ and any constants a , b .

b) Shift Property

$$\mathcal{F}[x(t-a)] = e^{-j2\pi fa} X(f) \quad (15)$$

c) Fourier Transform of Fourier Transform

$$\mathcal{F}[X(f)] = x(-t) \quad (16)$$

In words, the application of two successive Fourier transforms reverses the sign of the original independent variable.

d) Inverse Fourier Transforms of $X(f)$

$$x(t) = \mathcal{F}^{-1} [X(f)] = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \quad (17)$$

e) Even and Odd Function Properties

Let $X(f) = X_R(f) - j X_I(f)$

$$x(t) \text{ even} \Leftrightarrow X(f) = X_R(f) \text{ even}$$

$$x(t) \text{ odd} \Leftrightarrow X(f) = -j X_I(f) \text{ odd}$$

f) Similarity Property

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right) \quad (19)$$

g) Energy Property

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad (20)$$

This follows from Parseval's Theorem that

$$\int_{-\infty}^{\infty} x(t)y(t) dt = \int_{-\infty}^{\infty} X^*(f) Y(f) df$$

for any $x(t)$, $y(t)$ with Fourier transforms $X(f)$, $Y(f)$.

h) Autocorrelation Property

If the autocorrelation function of $x(t)$ is defined by the integral

$$R(t) = \int_{-\infty}^{\infty} x(u)x(u+t) du \quad (21)$$

then

$$\mathcal{F}[R(t)] = |X(f)|^2 \quad (22)$$

i) Product Property

$$\mathcal{F}[x(t)y(t)] = X(f) \star Y(f) \quad (23)$$

where $X(f) \star Y(f)$ is the convolution integral

$$X(f) \star Y(f) = \int_{-\infty}^{\infty} X(u) Y(f-u) du \quad (24)$$

In words, the Fourier transform of a product is the convolution of their transforms.

j) **Convolution Property**

$$\mathcal{F}[x(t) \star y(t)] = X(f) Y(f) \quad (25)$$

where $x(t) \star y(t)$ is the convolution integral

$$x(t) \star y(t) = \int_{-\infty}^{\infty} x(u) y(t - u) du \quad (26)$$

In words, the Fourier transform of a convolution is the product of their transforms.

3. HILBERT TRANSFORM FORMULAS

3.1. THREE DEFINITIONS FOR HILBERT TRANSFORM

(1) Definition as Convolution Integral

The **Hilbert transform** of a real-valued function $x(t)$ extending from $-\infty < t < \infty$ is a real-valued function $\tilde{x}(t)$ defined by

$$\tilde{x}(t) = \mathcal{H}[x(t)] = \int_{-\infty}^{\infty} \frac{x(u)}{\pi(t-u)} du \quad (27)$$

Thus $\tilde{x}(t)$ is the convolution integral of $x(t)$ with $(1/\pi t)$, written as

$$\tilde{x}(t) = x(t) \star (1/\pi t) \quad (28)$$

Like Fourier transforms, Hilbert transforms are linear operators where

$$\mathcal{H}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{H}[x_1(t)] + a_2 \mathcal{H}[x_2(t)] \quad (29)$$

for any constants a_1, a_2 and any functions $x_1(t), x_2(t)$.

(2) Definition as $(\pi/2)$ Phase Shift System

Let $\tilde{X}(f)$ be the Fourier transform of $\tilde{x}(t)$, namely,

$$\tilde{X}(f) = \mathcal{F}[\tilde{x}(t)] = \int_{-\infty}^{\infty} \tilde{x}(t) e^{-j2\pi ft} dt \quad (30)$$

Then, from Equations (25) and (28), it follows that $\tilde{X}(f)$ is the Fourier transform $X(f)$ of $x(t)$, multiplied by the Fourier transform of $(1/\pi t)$. The Fourier transform of $(1/\pi t)$ is given by

$$\mathcal{F}[1/\pi t] = -j \operatorname{sgn} f = \begin{cases} -j & \text{for } f > 0 \\ j & \text{for } f < 0 \end{cases} \quad (31)$$

Hence Equation (28) is equivalent to the passage of $x(t)$ through a system defined by $(-j \operatorname{sgn} f)$ to yield

$$\tilde{X}(f) = (-j \operatorname{sgn} f) X(f) \quad (32)$$

The complex-valued quantity $\tilde{X}(f)$ is the **Hilbert transform** of the complex-valued quantity $X(f)$ as defined by

$$\tilde{X}(f) = \mathcal{H}[X(f)] = (-j \operatorname{sgn} f)X(f) \quad (33)$$

Of course, $\tilde{X}(t)$ is the inverse Fourier transform of $\tilde{X}(f)$.

The Fourier transform $(-j \operatorname{sgn} f)$ can be represented by

$$B(f) = -j \operatorname{sgn} f = \begin{cases} e^{-j(\pi/2)} & \text{for } f > 0 \\ e^{j(\pi/2)} & \text{for } f < 0 \end{cases} \quad (34)$$

Also, as per Equation (6), one can write

$$B(f) = |B(f)| e^{-j\phi_b(f)} \quad (35)$$

Hence $B(f)$ is a $(\pi/2)$ phase shift system where

$$|B(f)| = 1 \quad \text{for all } f \quad (36)$$

$$\phi_b(f) = \begin{cases} \pi/2 & \text{for } f > 0 \\ -\pi/2 & \text{for } f < 0 \end{cases} \quad (37)$$

Again, as per Equation (6), if one writes

$$X(f) = |X(f)| e^{-j\phi_x(f)}$$

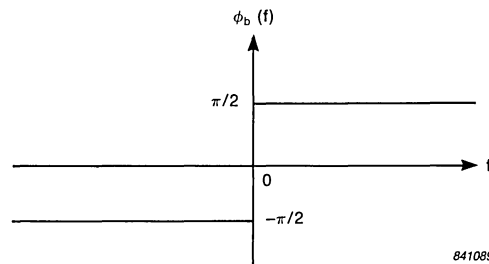
it follows that (38)

$$\tilde{X}(f) = |\tilde{X}(f)| e^{-j\tilde{\phi}_x(f)} = |X(f)| e^{-j[\phi_x(f) + \phi_b(f)]} \quad (39)$$

Thus the Hilbert transform consists of passing $X(t)$ through a system which leaves the magnitude of $X(f)$ unchanged, but changes the phase from $\phi_x(f)$ to $\phi_x(f) + \phi_b(f)$ using the $\phi_b(f)$ of Equation (37). Thus

$$\begin{aligned} \phi_x(f) &\rightarrow \phi_x(f) + (\pi/2) & \text{for } f > 0 \\ \phi_x(f) &\rightarrow \phi_x(f) - (\pi/2) & \text{for } f < 0 \end{aligned} \quad (40)$$

In words, shift $(\pi/2)$ for positive frequencies and shift $(-\pi/2)$ for negative frequencies.



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(3) Definition as Imaginary Part of Analytic Signal

A third useful way to understand and to compute the Hilbert transform $\tilde{x}(t)$ of $x(t)$ is to introduce the **analytic signal** $z(t)$ defined by

$$z(t) = x(t) + j \tilde{x}(t) \quad (41)$$

One can also write

$$z(t) = A(t)e^{j\phi(t)} \quad (42)$$

where $A(t)$ is called the **envelope signal** and $\phi(t)$ is called the **instantaneous phase signal**.

In terms of $x(t)$ and $\tilde{x}(t)$, the envelope $A(t)$ is the positive square root of $x^2(t) + \tilde{x}^2(t)$

$$A(t) = \sqrt{x^2(t) + \tilde{x}^2(t)} \quad (43)$$

and

$$\phi(t) = \tan^{-1}[\tilde{x}(t)/x(t)] \quad (44)$$

The “instantaneous frequency” f_o is given by

$$f_o = \left(\frac{1}{2\pi} \right) \frac{d\phi(t)}{dt} \quad (45)$$

Let $Z(f)$ be the Fourier transform of $z(t)$, namely,

$$\begin{aligned} Z(f) &= \mathcal{F}[z(t)] = \mathcal{F}[x(t) + j \tilde{x}(t)] \\ &= \mathcal{F}[x(t)] + j \mathcal{F}[\tilde{x}(t)] = X(f) + j \tilde{X}(f) \end{aligned} \quad (46)$$

The inverse Fourier transform of $Z(f)$ yields

$$z(t) = \mathcal{F}^{-1}[Z(f)] = x(t) + j \tilde{x}(t) \quad (47)$$

where

$$x(t) = \text{Real Part}[z(t)] \quad (48)$$

$$\tilde{x}(t) = \mathcal{H}[x(t)] = \text{Imag Part}[z(t)] \quad (49)$$

To compute $Z(f)$, note from Equation (32) that

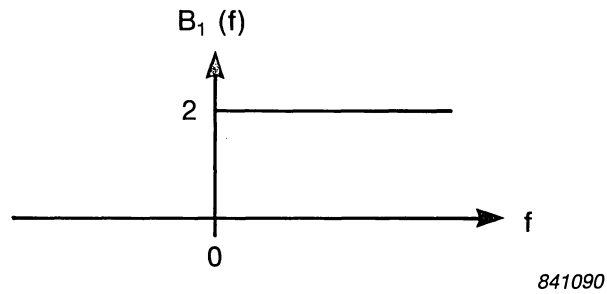
$$\tilde{X}(f) = (-j \operatorname{sgn} f) X(f)$$

Hence, Equation (46) becomes

$$Z(f) = [1 + \operatorname{sgn} f] X(f) = B_1(f) X(f) \quad (50)$$

where

$$B_1(f) = \begin{cases} 2 & \text{for } f > 0 \\ 0 & \text{for } f < 0 \end{cases} \quad (51)$$



This is a very simple transformation to obtain $Z(f)$ from $X(f)$.

One should compute $X(f)$ for all f and then define $Z(f)$ by

$$\begin{aligned} Z(f) &= 2X(f) & \text{for } f > 0 \\ &= 0 & \text{for } f < 0 \end{aligned} \quad (52)$$

The inverse Fourier transform of $Z(f)$ then gives $z(t)$ with $\tilde{x}(t) = \text{Imag}[z(t)]$.

3.2. EXAMPLES OF HILBERT TRANSFORMS

The following Table 2 gives several $x(t)$ with its associated $\tilde{x}(t)$ and envelope $A(t)$. Proofs of these results follow directly from previous definitions.

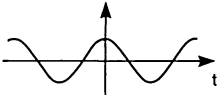
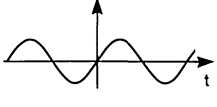
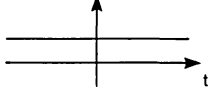
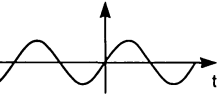
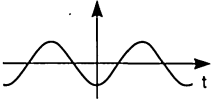
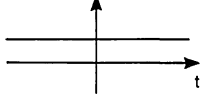
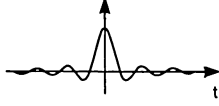
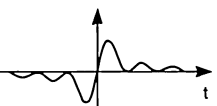
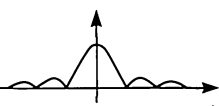
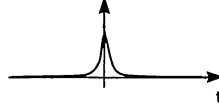
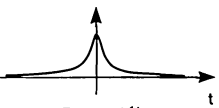
$x(t)$	$\tilde{x}(t)$	$A(t)$
 $\cos(2\pi f_0 t)$	 $\sin(2\pi f_0 t)$	 1
 $\sin(2\pi f_0 t)$	 $-\cos(2\pi f_0 t)$	 1
 $\frac{\sin t}{t}$	 $\frac{1 - \cos t}{t}$	 $\left \frac{\sin(t/2)}{(t/2)} \right $
 $\frac{1}{1 + t^2}$	 $\frac{t}{1 + t^2}$	 $\left[\frac{1}{1 + t^2} \right]^{1/2}$
 $e^{-c t } \cos(2\pi f_0 t)$	 $e^{-c t } \sin(2\pi f_0 t)$	 $e^{-c t }$

Table 2. Examples of Hilbert Transforms

The last result in the table is a special case of a general theorem proved by the author that

$$\mathcal{H}[u(t) \cos 2\pi f_0 t] = u(t) \sin 2\pi f_0 t$$

for any function $u(t)$ which is an even function.

3.3. PROPERTIES OF HILBERT TRANSFORMS

A number of properties will now be stated for Hilbert Transforms which follow readily from the basic definitions or are proved in References [4–6, 16]. Let $\tilde{x}(t) = \mathcal{H}[x(t)]$ and $\tilde{y}(t) = \mathcal{H}[y(t)]$ be Hilbert transforms of $x(t)$ and $y(t)$.

a) **Linear Property**

$$\mathcal{H}[ax(t) + b\tilde{y}(t)] = a\tilde{x}(t) + b\tilde{y}(t) \quad (53)$$

for any functions $x(t)$, $y(t)$ and any constants a , b .

b) **Shift Property**

$$\mathcal{H}[x(t - a)] = \tilde{x}(t - a) \quad (54)$$

c) **Hilbert Transform of Hilbert Transform**

$$\mathcal{H}[\tilde{x}(t)] = -x(t) \quad (55)$$

In words, the application of two successive Hilbert transforms gives the negative of the original function.

d) **Inverse Hilbert Transform of $\tilde{x}(t)$**

$$x(t) = \mathcal{H}^{-1}[\tilde{x}(t)] = - \int_{-\infty}^{\infty} \frac{\tilde{x}(u)}{\pi(t - u)} du \quad (56)$$

Thus $x(t)$ is the convolution of $\tilde{x}(t)$ with $(-1/\pi t)$. Alternatively, $x(t)$ can be defined by

$$x(t) = \mathcal{F}^{-1}[j \operatorname{sgn} f \tilde{X}(f)] \quad (57)$$

where

$$\tilde{X}(f) = \mathcal{F}[\tilde{x}(t)]$$

e) **Even and Odd Function Properties**

If $x(t)$ is an even (odd) function of t , then $\tilde{x}(t)$ is an odd (even) function of t .

$$x(t) \text{ even} \leftrightarrow \tilde{x}(t) \text{ odd}$$

$$x(t) \text{ odd} \leftrightarrow \tilde{x}(t) \text{ even} \quad (58)$$

f) **Similarity Property**

$$\mathcal{H}[x(at)] = \tilde{x}(at) \quad (59)$$

g) **Energy Property**

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} \tilde{x}^2(t) dt \quad (60)$$

This follows from Parseval's Theorem since

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad (61)$$

$$\int_{-\infty}^{\infty} \tilde{x}^2(t) dt = \int_{-\infty}^{\infty} |\tilde{X}(f)|^2 df$$

and the fact that

$$|\tilde{X}(f)|^2 = |X(f)|^2 \quad (62)$$

h) Orthogonal Property

$$\int_{-\infty}^{\infty} x(t)\tilde{x}(t) dt = 0 \quad (63)$$

This follows from Parseval's Theorem since

$$\int_{-\infty}^{\infty} x(t)\tilde{x}(t) dt = \int_{-\infty}^{\infty} X^*(f)\tilde{X}(f) df \quad (64)$$

and the fact that

$$X^*(f)\tilde{X}(f) = (-j \operatorname{sgn} f) |X(f)|^2$$

is an odd function of f so that the right-hand side of Equation (64) is zero.

i) Modulation Property

$$\mathcal{H}[x(t) \cos 2\pi f_o t] = x(t) \sin 2\pi f_o t \quad (65)$$

for any $x(t)$ whose Fourier transform $X(f)$ is band-limited, that is,

$$X(f) = \begin{cases} X(f) & |f| \leq F \\ 0 & \text{otherwise} \end{cases} \quad (66)$$

provided $f_o > F$. Also

$$\mathcal{H}[x(t) \sin 2\pi f_o t] = -x(t) \cos 2\pi f_o t \quad (67)$$

j) Convolution Property

$$\mathcal{H}[x(t) * y(t)] = \tilde{x}(t) * y(t) = x(t) * \tilde{y}(t) \quad (68)$$

This follows from the fact that

$$\mathcal{F}[x(t) * y(t)] = X(f) Y(f) \quad (69)$$

and

$$\begin{aligned} [(-j \operatorname{sgn} f) X(f)] Y(f) &= \tilde{X}(f) Y(f) \\ &= X(f) [(-j \operatorname{sgn} f) Y(f)] = X(f) \tilde{Y}(f) \end{aligned} \quad (70)$$

4. HILBERT TRANSFORMS FOR CORRELATION FUNCTIONS OF STATIONARY RANDOM DATA

4.1. CORRELATION AND ENVELOPE DEFINITIONS

Let $x(t)$ and $y(t)$ represent zero mean value stationary random data with autocorrelation functions $R_{xx}(\tau)$ and $R_{yy}(\tau)$ and cross-correlation function $R_{xy}(\tau)$. Let the associated two-sided autospectral density functions be $S_{xx}(f)$ and $S_{yy}(f)$, and the associated two-sided cross-spectral density function be $S_{xy}(f)$. As proved in References [2, 3], such associated functions are Fourier transform pairs. For each stationary random record $x(t)$ or $y(t)$, let $\tilde{x}(t) = \mathcal{H}[x(t)]$ and $\tilde{y}(t) = \mathcal{H}[y(t)]$ be their Hilbert transforms. Since Hilbert transforms are linear operations, it follows that $\tilde{x}(t)$ and $\tilde{y}(t)$ will also be stationary random data. Various correlation and spectral density functions for $\tilde{x}(t)$ and $\tilde{y}(t)$ can be defined the same as for $x(t)$ and $y(t)$ to yield $R_{\tilde{x}\tilde{x}}(\tau)$, $R_{\tilde{y}\tilde{y}}(\tau)$, $R_{\tilde{x}\tilde{y}}(\tau)$ and $S_{\tilde{x}\tilde{x}}(f)$, $S_{\tilde{y}\tilde{y}}(f)$, $S_{\tilde{x}\tilde{y}}(f)$ where associated functions are Fourier transform pairs.

To be specific, the definitions are

$$\begin{aligned} R_{\tilde{x}\tilde{x}}(\tau) &= E[\tilde{x}(t)\tilde{x}(t + \tau)] \\ R_{\tilde{y}\tilde{y}}(\tau) &= E[\tilde{y}(t)\tilde{y}(t + \tau)] \\ R_{\tilde{x}\tilde{y}}(\tau) &= E[\tilde{x}(t)\tilde{y}(t + \tau)] \end{aligned} \tag{71}$$

where $E[\]$ is the expected value of the quantity within the brackets. Then

$$\begin{aligned} S_{\tilde{x}\tilde{x}}(f) &= \mathcal{F}[R_{\tilde{x}\tilde{x}}(\tau)] \\ S_{\tilde{y}\tilde{y}}(f) &= \mathcal{F}[R_{\tilde{y}\tilde{y}}(\tau)] \\ S_{\tilde{x}\tilde{y}}(f) &= \mathcal{F}[R_{\tilde{x}\tilde{y}}(\tau)] \end{aligned} \tag{72}$$

Except for a scale factor (which is unimportant for theoretical derivations since it will appear on both sides of equations and therefore cancel out), one can also use the definitions

$$\begin{aligned} S_{\tilde{x}\tilde{x}}(f) &= E[\tilde{X}^*(f)\tilde{X}(f)] = E[|\tilde{X}(f)|^2] \\ S_{\tilde{y}\tilde{y}}(f) &= E[\tilde{Y}^*(f)\tilde{Y}(f)] = E[|\tilde{Y}(f)|^2] \\ S_{\tilde{x}\tilde{y}}(f) &= E[\tilde{X}^*(f)\tilde{Y}(f)] \end{aligned} \tag{73}$$

Mixing functions and their Hilbert transforms leads to $R_{x\tilde{y}}(\tau)$, $R_{\tilde{x}y}(\tau)$ and associated $S_{x\tilde{y}}(f)$, $S_{\tilde{x}y}(f)$ defined by

$$R_{x\tilde{y}}(\tau) = E[x(t)\tilde{y}(t + \tau)] \quad (74)$$

$$R_{\tilde{x}y}(\tau) = E[\tilde{x}(t)y(t + \tau)]$$

$$S_{x\tilde{y}}(f) = \mathcal{F} [R_{x\tilde{y}}(\tau)] \quad (75)$$

$$S_{\tilde{x}y}(f) = \mathcal{F} [R_{\tilde{x}y}(\tau)]$$

Special cases are $R_{x\tilde{x}}(\tau)$, $R_{\tilde{x}x}(\tau)$, $S_{x\tilde{x}}(f)$ and $S_{\tilde{x}x}(f)$. Except for a scale factor, as noted previously, one can also define $S_{x\tilde{y}}(f)$ and $S_{\tilde{x}y}(f)$ directly by

$$S_{x\tilde{y}}(f) = E[X^*(f)\tilde{Y}(f)] \quad (76)$$

$$S_{\tilde{x}y}(f) = E[\tilde{X}^*(f)Y(f)]$$

Envelope functions for $R_{xx}(\tau)$, $R_{yy}(\tau)$ and $R_{xy}(\tau)$ are defined as follows in terms of Hilbert transforms $\tilde{R}_{xx}(\tau)$, $\tilde{R}_{yy}(\tau)$ and $\tilde{R}_{xy}(\tau)$.

The envelope function for $R_{xx}(\tau)$ is the positive square root

$$A_{xx}(\tau) = [R_{xx}^2(\tau) + \tilde{R}_{xx}^2(\tau)]^{1/2} \quad (77)$$

The envelope function for $R_{yy}(\tau)$ is the positive square root

$$A_{yy}(\tau) = [R_{yy}^2(\tau) + \tilde{R}_{yy}^2(\tau)]^{1/2} \quad (78)$$

The envelope function for $R_{xy}(\tau)$ is the positive square root

$$A_{xy}(\tau) = [R_{xy}^2(\tau) + \tilde{R}_{xy}^2(\tau)]^{1/2} \quad (79)$$

Various relationships and properties for all of the above quantities will now be stated.

4.2. HILBERT TRANSFORM RELATIONSHIPS

The following Table 3 gives a number of useful relationships for autocorrelation (autospectral density) and cross-correlation (cross-spectral density) functions. Proofs of these results represent straight-forward exercises.

$R_{\tilde{x}\tilde{x}}(\tau) = R_{xx}(\tau)$	$S_{\tilde{x}\tilde{x}}(f) = S_{xx}(f)$
$R_{\tilde{x}\tilde{x}}(-\tau) = R_{\tilde{x}\tilde{x}}(\tau)$, an even function of τ	
$\sigma_{\tilde{x}}^2 = R_{\tilde{x}\tilde{x}}(0) = R_{xx}(0) = \sigma_x^2$	
$\tilde{R}_{xx}(\tau) = R_{\tilde{x}\tilde{x}}(\tau) = -R_{\tilde{x}x}(\tau)$	$\tilde{S}_{xx}(f) = S_{\tilde{x}\tilde{x}}(f) = -S_{\tilde{x}x}(f)$
$\tilde{R}_{xx}(-\tau) = -\tilde{R}_{xx}(\tau)$, an odd function of τ	
$\tilde{R}_{xx}(0) = R_{\tilde{x}\tilde{x}}(0) = 0$	
$R_{\tilde{x}\tilde{y}}(\tau) = R_{xy}(\tau)$	$S_{\tilde{x}\tilde{y}}(f) = S_{xy}(f)$
$\sigma_{\tilde{x}\tilde{y}} = R_{\tilde{x}\tilde{y}}(0) = R_{xy}(0) = \sigma_{xy}$	
$\tilde{R}_{xy}(\tau) = R_{\tilde{x}\tilde{y}}(\tau) = -R_{\tilde{x}y}(\tau)$	$\tilde{S}_{xy}(f) = S_{\tilde{x}\tilde{y}}(f) = -S_{\tilde{x}y}(f)$
$\tilde{R}_{xy}(-\tau) = R_{\tilde{y}x}(\tau)$	$\tilde{S}_{xy}^*(f) = S_{\tilde{y}x}(f)$

Table 3. Hilbert Transform Relationships

4.3. ANALYTIC SIGNALS FOR CORRELATION FUNCTIONS

The analytic signal $z_x(t)$ corresponding to a stationary random record $x(t)$ is defined as in Equation (41) by

$$z_x(t) = x(t) + j\tilde{x}(t) \quad (80)$$

where $\tilde{x}(t) = \mathcal{H}[x(t)]$. The complex-valued autocorrelation function $R_{z_x z_x}(\tau)$ is defined by

$$\begin{aligned} R_{z_x z_x}(\tau) &= E[z_x^*(t) z_x(t + \tau)] = E[\{x(t) - j\tilde{x}(t)\} \{x(t + \tau) + j\tilde{x}(t + \tau)\}] \\ &= R_{xx}(\tau) + R_{\tilde{x}\tilde{x}}(\tau) + j[R_{x\tilde{x}}(\tau) - R_{\tilde{x}x}(\tau)] \end{aligned}$$

But

$$R_{\tilde{x}\tilde{x}}(\tau) = R_{xx}(\tau) \quad \text{and} \quad R_{\tilde{x}x}(\tau) = -R_{x\tilde{x}}(\tau) = -\tilde{R}_{xx}(\tau)$$

Hence

$$R_{z_x z_x}(\tau) = 2[R_{xx}(\tau) + j\tilde{R}_{xx}(\tau)] = 2[R_{xx}(\tau) + j\tilde{R}_{xx}(\tau)] \quad (81)$$

This proves that

$$\frac{R_{z_x z_x}(\tau)}{2} = R_{xx}(\tau) + j\tilde{R}_{xx}(\tau) \quad (82)$$

is the analytic signal for $R_{xx}(\tau)$.

The Fourier transform of $R_{z_X z_X}(\tau)$ is $S_{z_X z_X}(f)$ where

$$\begin{aligned} S_{z_X z_X}(f) &= \mathcal{F} [R_{z_X z_X}(\tau)] = 2 [S_{xx}(f) + j\tilde{S}_{xx}(f)] \\ &= 2 [1 + \operatorname{sgn} f] S_{xx}(f) \end{aligned} \quad (83)$$

Hence

$$S_{z_X z_X}(f) = \begin{cases} 4S_{xx}(f) & \text{for } f > 0 \\ 0 & \text{for } f < 0 \end{cases} \quad (84)$$

From Equation (82), one then obtains

$$\mathcal{F}^{-1} [S_{z_X z_X}(f)/2] = R_{xx}(\tau) + j\tilde{R}_{xx}(\tau) \quad (85)$$

Theoretical formulas are now

$$R_{xx}(\tau) = 2 \int_0^\infty S_{xx}(f) \cos 2\pi f \tau \, df \quad (86)$$

$$\tilde{R}_{xx}(\tau) = 2 \int_0^\infty S_{xx}(f) \sin 2\pi f \tau \, df \quad (87)$$

The analytic signal for $R_{xy}(\tau)$ corresponding to stationary random records $x(t)$ and $y(t)$ is defined from

$$z_x(t) = x(t) + j\tilde{x}(t) \quad z_y(t) = y(t) + j\tilde{y}(t) \quad (88)$$

The complex-valued cross-correlation function $R_{z_X z_Y}(\tau)$ is then defined by

$$\begin{aligned} R_{z_X z_Y}(\tau) &= E [z_x^*(t) z_y(t + \tau)] = E [\{ x(t) - j\tilde{x}(t) \} \{ y(t + \tau) + j\tilde{y}(t + \tau) \}] \\ &= R_{xy}(\tau) + R_{\tilde{x}\tilde{y}}(\tau) + j [R_{x\tilde{y}}(\tau) - R_{\tilde{x}y}(\tau)] \end{aligned}$$

But

$$R_{\tilde{x}\tilde{y}}(\tau) = R_{xy}(\tau) \quad \text{and} \quad R_{\tilde{x}y}(\tau) = -R_{xy}(\tau) = -\tilde{R}_{xy}(\tau)$$

Hence

$$R_{z_X z_Y}(\tau) = 2 [R_{xy}(\tau) + j\tilde{R}_{xy}(\tau)] = 2 [R_{xy}(\tau) + j\tilde{R}_{xy}(\tau)] \quad (89)$$

This proves

$$\frac{R_{z_X z_Y}(\tau)}{2} = R_{xy}(\tau) + j\tilde{R}_{xy}(\tau) \quad (90)$$

is the analytic signal for $R_{xy}(\tau)$.

The Fourier transform of $R_{z_x z_y}(\tau)$ is $S_{z_x z_y}(f)$ where

$$\begin{aligned} S_{z_x z_y}(f) &= \mathcal{F} [R_{z_x z_y}(\tau)] = 2 [S_{xy}(f) + j\tilde{S}_{xy}(f)] \\ &= 2 [1 + \text{sgn}f] S_{xy}(f) \end{aligned} \quad (91)$$

Hence

$$S_{z_x z_y}(f) = \begin{cases} 4 S_{xy}(f) & \text{for } f > 0 \\ 0 & \text{for } f < 0 \end{cases} \quad (92)$$

From Equation (88), one then obtains

$$\mathcal{F}^{-1} [S_{z_x z_y}(f)/2] = R_{xy}(\tau) + j\tilde{R}_{xy}(\tau) \quad (93)$$

This yields the theoretical formulas

$$R_{xy}(\tau) = \text{Real} \left[2 \int_0^\infty S_{xy}(f) e^{j2\pi f\tau} df \right] \quad (94)$$

$$\tilde{R}_{xy}(\tau) = \text{Imag} \left[2 \int_0^\infty S_{xy}(f) e^{j2\pi f\tau} df \right] \quad (95)$$

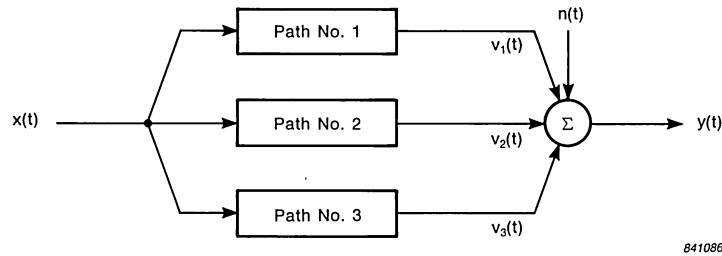
For digital processing of finite length records, there are additional requirements to avoid “circular” correlation functions and biasing effects. As discussed in References [2, 3], zero padding is recommended together with appropriate scale factors to retain overall mean square values and remove a “bow-tie” bias. Shifting of records may also be required to remove time delay bias effects.

5. NONDISPERSIVE PROPAGATION PROBLEMS

Three important application areas of Hilbert transforms for correlation functions occur in nondispersive propagation input/output problems such as the following Cases 1, 2 and 3.

5.1. THREE DIFFERENT CASES

Case 1. Nondispersive Propagation Through Multiple Paths

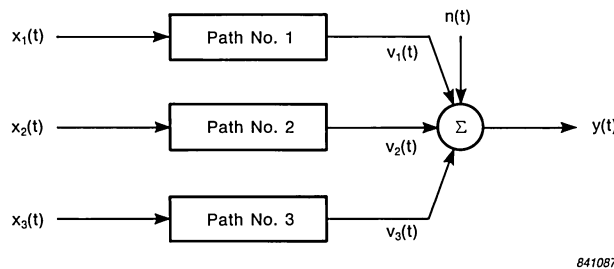


For simplicity, only three paths are shown but the number is arbitrary. The governing equation for this model is

$$y(t) = \alpha_1 x(t - \tau_1) + \alpha_2 x(t - \tau_2) + \alpha_3 x(t - \tau_3) + n(t)$$

where α_i are constant attenuation factors for each path and τ_i are the respective time delays for each path. The noise $n(t)$ is assumed to be uncorrelated with $x(t)$. Simultaneous measurements are made only of $x(t)$ and $y(t)$. It is required to determine the proportion of the output power in $y(t)$ that goes via each path. Cross-correlation measurements are made between $x(t)$ and $y(t)$.

Case 2. Nondispersive Propagation from Multiple Uncorrelated Sources

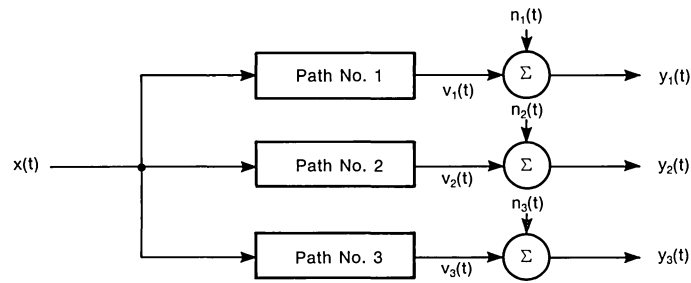


Simultaneous measurements are made of all inputs and the output. For simplicity, only three sources and associated paths are shown but the number of uncorrelated sources can be arbitrary. It is required to determine the proportion of the output power in $y(t)$ that comes from each source, and the respective time delays for passage of each source through its particular path. The output noise $n(t)$ is assumed to be uncorrelated with the inputs $x_i(t)$, and the three sources are also assumed to be uncorrelated with each other. The governing equation for this model is

$$y(t) = \alpha_1 x_1(t - \tau_1) + \alpha_2 x_2(t - \tau_2) + \alpha_3 x_3(t - \tau_3) + n(t)$$

where α_i are constant attenuation factors for each path and τ_i are the respective time delays for each path. Cross-correlation measurements are made between each $x_i(t)$ and $y(t)$.

Case 3. Nondispersive Propagation From an Unmeasured Single Source to Measured Multiple Outputs



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Simultaneous measurements are made of all output signals. For simplicity, only three outputs are shown but the number of different outputs can be arbitrary. It is required to determine the relative time delay between all pairs of output signals without knowing the actual time delays from the input $x(t)$ to any of the outputs, assuming $x(t)$ is not or cannot be measured. The extraneous noise terms $n_i(t)$ are all assumed to be uncorrelated with each other and with $x(t)$. The governing equations for this model are that each output for $i = 1, 2, 3$ is of the form

$$y_i(t) = v_i(t) + n_i(t) = \alpha_i x(t - \tau_i) + n_i(t)$$

Cross-correlation measurements are made between pairs of output records.

5.2. BASIC NONDISPERSIVE PROPAGATION MODEL

The nondispersive propagation problems of Cases 1, 2 and 3 all involve cross-correlation measurements that are variations of the basic model pictured in Figure 1. Let the output response $r(t)$ at point B be a delayed version of an input signal $s(t)$ at point A. There is a positive time delay $\tau_0 = (d/v)$ in going from A to B where d is the distance and v is the constant velocity of propagation that is independent of frequency.

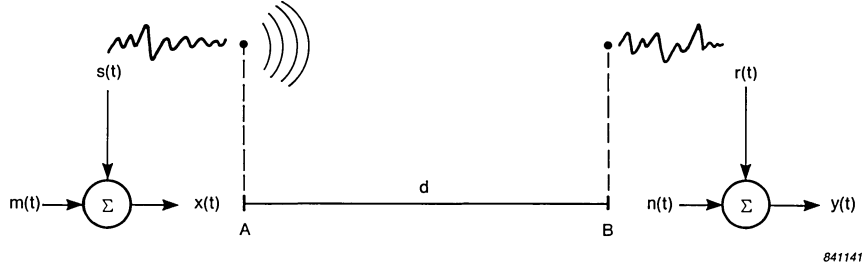


Fig. 1. Nondispersive Propagation Model

Under ideal conditions, for a single path,

$$r(t) = \alpha s(t - \tau_0) \quad (96)$$

where α is a constant factor.

Observe that $r(\tau_0) = \alpha s(0)$ and that $s(t)$ for $0 \leq t \leq T$ corresponds to $r(t)$ for $\tau_0 \leq t \leq \tau_0 + T$.

Let $x(t)$ and $y(t)$ be measurements of $s(t)$ and $r(t)$ respectively, where

$$x(t) = s(t) + m(t) \quad (97)$$

$$y(t) = r(t) + n(t) = \alpha s(t - \tau_0) + n(t) \quad (98)$$

The quantities $m(t)$ and $n(t)$ are extraneous noise terms assumed to be uncorrelated with each other and with $s(t)$. For stationary random data, the cross-correlation function

$$R_{xy}(\tau) = E[x(t)y(t + \tau)] = \alpha R_{ss}(\tau - \tau_0) \quad (99)$$

where $E[\quad]$ denotes the expected value operation and where $R_{ss}(\tau)$ is the autocorrelation function of $s(t)$. Hence, for this model, $R_{xy}(\tau)$ will have a peak value at $\tau = \tau_0$ since

$$R_{xy}(\tau_0) = \alpha R_{ss}(0) > \alpha |R_{ss}(\tau)| \text{ for all } \tau \quad (100)$$

From Equations (97) and (98), one obtains

$$R_{xx}(\tau) = R_{ss}(\tau) + R_{mm}(\tau) \quad (101)$$

$$R_{yy}(\tau) = \alpha^2 R_{ss}(\tau) + R_{nn}(\tau) \quad (102)$$

Hence, the **cross-correlation coefficient function**

$$\rho_{xy}(\tau) = \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}} = \frac{\alpha R_{ss}(\tau - \tau_0)}{\sigma_x \sigma_y} \quad (103)$$

where assuming zero mean values for $s(t)$, $m(t)$ and $n(t)$,

$$\begin{aligned}\sigma_x^2 &= R_{xx}(0) = \sigma_s^2 + \sigma_m^2 \\ \sigma_y^2 &= R_{yy}(0) = \alpha^2 \sigma_s^2 + \sigma_n^2\end{aligned}\tag{104}$$

Finally, at $\tau = \tau_o$ where $\rho_{xy}(\tau)$ assumes its maximum value

$$\rho_{xy}(\tau_o) = \frac{\alpha R_{ss}(0)}{\sigma_x \sigma_y}\tag{106}$$

This result can be expressed in terms of signal-to-noise ratios (σ_m/σ_s) and (σ_n/σ_s) .

When $\sigma_m = 0$, meaning negligible input noise, then $x(t) = s(t)$ and $R_{ss}(0) = \sigma_x^2$ so that

$$\rho_{xy}(\tau_o) = \frac{\alpha \sigma_x}{\sigma_y}\tag{107}$$

Now

$$\alpha = \rho_{xy}(\tau_o) \left(\frac{\sigma_y}{\sigma_x} \right)\tag{108}$$

For this value of α , the quantity

$$R_{yy}(0) = R_{y:x}(0) + R_{y \cdot x}(0)\tag{109}$$

where

$$R_{y:x}(0) = \alpha^2 \sigma_x^2 = \rho_{xy}^2(\tau_o) \sigma_y^2\tag{110}$$

represents the variance in $y(t)$ due to $x(t)$. For $\sigma_m \neq 0$, the above relations do not hold of Equations (107) to (110).

In practice, for arbitrary values of σ_m , it is desired to determine the peak value $\rho_{xy}(\tau_o)$ of Equation (106) and the precise location $\tau = \tau_o$ where this peak occurs. This peak value occurs at the value of τ where $R_{ss}(\tau)$ passes through $\tau = 0$. See Figure 2.

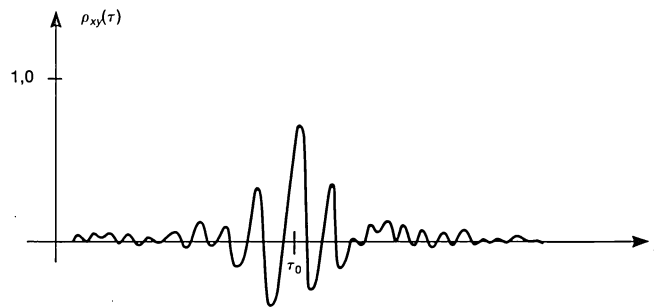


Fig. 2. Typical Plot of $\rho_{xy}(\tau)$ versus τ

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As this plot shows, it is often difficult to measure the peak value and to find just where the peak occurs due to the fine structure in the function $\rho_{xy}(\tau)$. What is really required is the peak value and the location where the peak occurs in the envelope function for $\rho_{xy}(\tau)$. Equation (103) shows that except for a scale factor of $(\alpha/\sigma_x\sigma_y)$, one wants to measure the envelope function of $R_{xy}(\tau)$.

The Hilbert transform of $R_{xy}(\tau)$ can be used to compute the desired envelope function. Important properties for such Hilbert transforms are covered in this paper as needed for this application as well as many other applications.

5.3. BAND-LIMITED AND NARROW-BAND WHITE NOISE

In order to develop some theoretical guidelines, consider cases of band-limited white noise and narrow-band white noise as pictured in Figures 3 and 4.

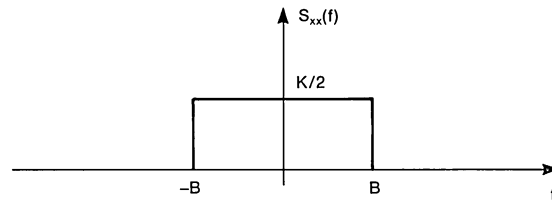
Band-Limited White Noise

For this case, the two-sided autospectral density function

$$S_{xx}(f) = \begin{cases} K/2 & |f| \leq B \\ 0 & \text{otherwise} \end{cases} \quad (111)$$

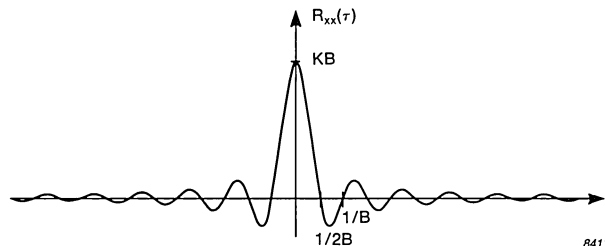
The associated autocorrelation function with main lobe extending from $(-1/2B)$ to $(1/2B)$ is given by

$$R_{xx}(\tau) = KB \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right) \quad (112)$$



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Fig. 3a. Autospectra for Band-limited White Noise



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Fig. 3b. Autocorrelation for Band-limited White Noise

Narrow-Band White Noise

For this case, the two-sided autospectral density function

$$S_{xx}(f) = \begin{cases} K/2 & |f \pm f_o| \leq (B_o/2) \\ 0 & \text{otherwise} \end{cases} \quad (113)$$

The bandwidth B_o here and B in Equation (111) are arbitrary. In practice one usually has $B_o < f_o$. The associated autocorrelation function

$$R_{xx}(\tau) = KB_o \left(\frac{\sin \pi B_o \tau}{\pi B_o \tau} \right) \cos 2\pi f_o \tau \quad (114)$$

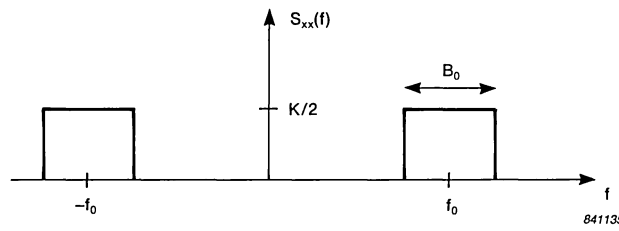


Fig. 4a. Autospectra for Narrow-Band White Noise

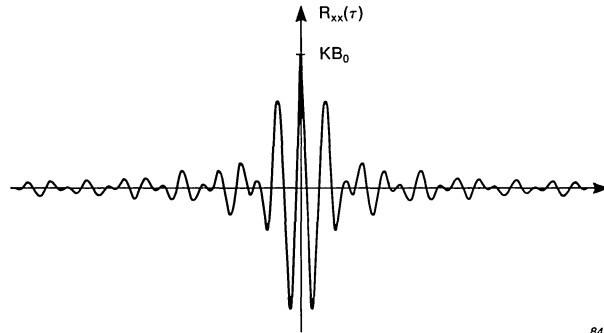


Fig. 4b. Autocorrelation for Narrow-Band White Noise

For band-limited white noise, $\tilde{R}_{xx}(\tau)$ and the envelope $A_{xx}(\tau)$ are given by, see Figure 5,

$$\tilde{R}_{xx}(\tau) = KB \left[\frac{1 - \cos 2\pi B\tau}{2\pi B\tau} \right] \quad (115)$$

$$A_{xx}(\tau) = KB \left| \frac{\sin \pi B\tau}{\pi B\tau} \right| \quad (116)$$

The width of the main lobe for $A_{xx}(\tau)$ extends from $(-1/B)$ to $(1/B)$.

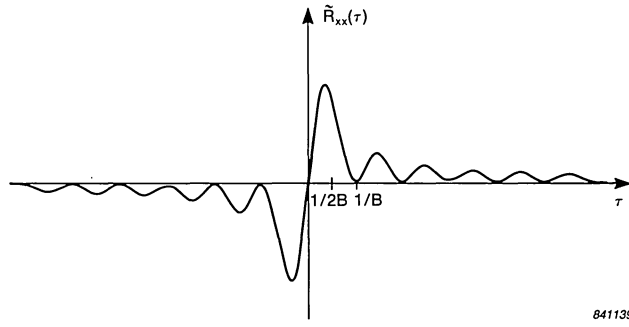


Fig. 5a. Hilbert Transform of Fig.3b

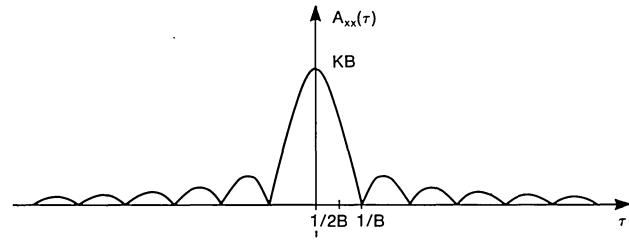


Fig. 5b. Envelope for Fig.3b

For narrow-band white noise, the results for $\tilde{R}_{xx}(\tau)$ and the envelope $A_{xx}(\tau)$ are, see Figure 6,

$$\tilde{R}_{xx}(\tau) = KB_o \left(\frac{\sin \pi B_o \tau}{\pi B_o \tau} \right) \sin 2\pi f_o \tau \quad (117)$$

$$A_{xx}(\tau) = KB_o \left| \frac{\sin \pi B_o \tau}{\pi B_o \tau} \right| \quad (118)$$

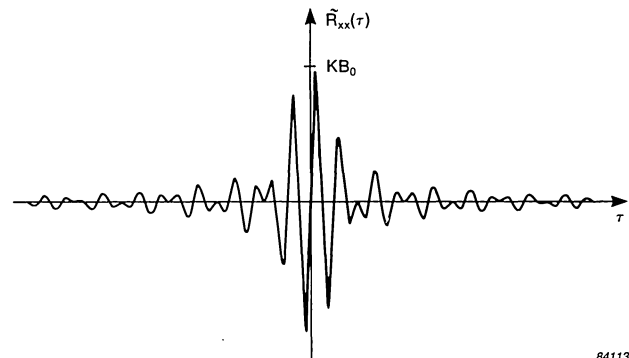


Fig. 6a. Hilbert Transform of Fig.4b

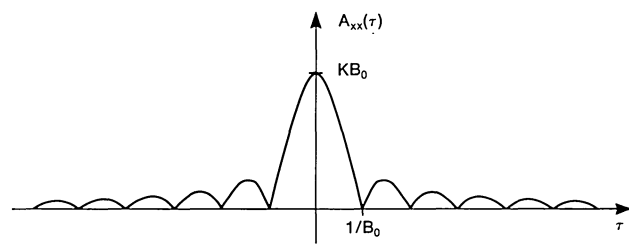


Fig. 6b. Envelope of Fig.4b

Note that band-limited white noise and narrow-band white noise have different auto-correlation functions and different Hilbert transforms, but they have similar envelope functions.

Note that in both of these cases, as well as for other cases, the maximum values of $R_{xx}(\tau)$ and $A_{xx}(\tau)$ occur at $\tau = 0$, namely for all τ ,

$$R_{xx}(0) > |R_{xx}(\tau)| \text{ and } A_{xx}(0) > |A_{xx}(\tau)| \quad (119)$$

Note also that at $\tau = 0$, the maximum value

$$R_{xx}(0) = A_{xx}(0) \quad (120)$$

and the Hilbert transform

$$\tilde{R}_{xx}(0) = 0 \quad (121)$$

where the crossing of zero by $\tilde{R}_{xx}(\tau)$ will be with positive slope, that is

$$\tilde{R}_{xx}(0+) > 0 \quad \text{and} \quad \tilde{R}_{xx}(0-) < 0 \quad (122)$$

In practice, it may often be considerably easier to find the location where zero crossings of $\tilde{R}_{xx}(\tau)$ occur than the location where maximum values of $R_{xx}(\tau)$ occur. The precise location where maximum values of $A_{xx}(\tau)$ occur may also be easier to find than the location where maximum values of $R_{xx}(\tau)$ occur. Besides location, determination of the peak magnitude values $R_{xx}(0)$ and $A_{xx}(0)$ represent two other important measurements.

6. STATISTICAL ERRORS IN MAGNITUDE ESTIMATES

Application of Hilbert transform techniques to determine peak values and associated time delays for correlation functions in nondispersive propagation problems requires knowledge of statistical error analysis criteria to design experiments and to evaluate estimates of computed results. In the following material, Section 6 contains appropriate statistical error analysis formulas for peak magnitude estimates of cross-correlation estimates, their Hilbert transform estimates and for the associated envelope estimates. Section 7 develops the statistical uncertainty in locating the position (time delay) where peak values in cross-correlation functions occur, and related statistical uncertainties in locating the position of corresponding zero crossings in their Hilbert transforms and peaks in the associated envelope estimates.

6.1. CORRELATION FUNCTION ESTIMATES

Consider the nondispersive propagation model of Figure 1. Assume that $x(t)$ and $y(t)$ behave like Gaussian band-limited white noise of bandwidth B . [Similar results apply for Gaussian narrow-band white noise.] Assume also that $x(t)$ and $y(t)$ are finite records of total duration T_{total} which can be divided into n_d disjoint (distinct, different) records of length T each such that $T_{total} = n_d T$. For such data, it is proved in [1, 2] that the normalized random error in the cross-correlation function estimate $\hat{R}_{xy}(\tau)$ is given approximately by

$$\epsilon \left[\hat{R}_{xy}(\tau) \right] \approx \frac{[1 + \rho_{xy}^{-2}(\tau)]}{[2BT_{total}]^{1/2}} \quad (123)$$

where $\rho_{xy}(\tau)$ is the cross-correlation coefficient function of Equation (103). At $\tau = \tau_o$ where $\rho_{xy}(\tau)$ assumes its maximum value, $\rho_{xy}(\tau_o)$ is given by Equation (106).

By assuming values for unknown terms such as $\rho_{xy}(\tau_o)$ and B , Equation (123) can be applied to determine T_{total} for design of experimental programs to achieve desired values of ϵ . After the experiments are performed, this same formula can calculate ϵ for estimates $\hat{R}_{xy}(\tau_o)$ and B together with knowledge of the available T_{total} . If ϵ is small, say $\epsilon \leq 0.10$, then one can state with 95% confidence that the unknown true value of $R_{xy}(\tau_o)$ lies in the interval

$$\left[(1 - 2\epsilon) \hat{R}_{xy}(\tau_o) \leq R_{xy}(\tau_o) \leq (1 + 2\epsilon) \hat{R}_{xy}(\tau_o) \right] \quad (124)$$

where ϵ here represents

$$\epsilon = \frac{\sigma[\hat{R}_{xy}(\tau_o)]}{R_{xy}(\tau_o)} \approx \frac{[1 + \hat{\rho}_{xy}^{-2}(\tau_o)]^{1/2}}{[2BT_{total}]^{1/2}} \quad (125)$$

The bandwidth-limited white noise bandwidth B in the error analysis formulas requires special understanding. Reference [17] shows that this noise bandwidth can be defined for zero mean value stationary random processes by

$$B = \frac{\int_0^\infty G_{yy}(f) df}{G_{yy}(f) |_{\max}} = \frac{R_{yy}(0)}{G_{yy}(f) |_{\max}} = \frac{\sigma_y^2}{G_{yy}(f) |_{\max}} \quad (126)$$

In digital processing of data, if the bandwidth resolution between spectral lines is Δf , then $\Delta f = (1/T)$ when T_{total} is assumed to be divided into n_d disjoint parts of length T each such that $T_{total} = n_d T$. Hence

$$n_d = \frac{T_{total}}{T} = (T_{total}) \Delta f \quad (127)$$

The maximum value for B is f_c , the Nyquist cutoff frequency. When this occurs

$$B = f_c = (N/2) \Delta f \quad (128)$$

where $f_c = (1/2\Delta t)$ and $T = N\Delta t = (1/\Delta f)$ using parameters defined in References [2, 3]. Thus, for digital processing one can replace $2BT_{total}$ in the error formulas by

$$2BT_{total} = Nn_d \quad (129)$$

In particular, at the peak value $\hat{R}_{xy}(\tau_o)$, Equation (125) becomes

$$\epsilon = \epsilon[\hat{R}_{xy}(\tau_o)] \approx \frac{[1 + \hat{\rho}_{xy}^2(\tau_o)]^{1/2}}{(Nn_d)^{1/2}} \quad (130)$$

It is straightforward to compute ϵ as a function of Nn_d for various values of $\rho_{xy}(\tau_o)$. Table 4 lists approximate values that must be satisfied between $\rho_{xy}(\tau_o)$ and Nn_d to achieve $\epsilon = 0.02$.

$\rho_{xy}(\tau_o)$	0.20	0.40	0.60	0.80	1.00
Nn_d	65,000	18,125	9,444	6,406	5,000

Table 4. Conditions for $\epsilon = 0.02$

To illustrate application of Table 4, suppose $\rho_{xy}(\tau_o) = 0.60$ and $Nn_d = 9,444$ from say $N = 500$ and $n_d = 19$, or from the values $B = 50$ Hz and $T_{total} = 95$ seconds. Then $\epsilon = 0.02$ and the 95% confidence interval for the peak value $R_{xy}(\tau_o)$ will be the interval from $0.96 \hat{R}_{xy}(\tau_o)$ to $1.04 \hat{R}_{xy}(\tau_o)$. As another example, for an experiment to achieve an $\epsilon = 0.02$, suppose it is assumed that $\rho_{xy}(\tau_o) = 0.20$ and that $B = f_c = 100$ Hz. If one requires $\Delta f = 0.50$ Hz, then $N = 400$ and $T = 2$ seconds. Table 4 shows that one should have $Nn_d = 65,000$. Hence n_d should be $n_d = 163$ and $T_{total} = 326$ seconds.

6.2. HILBERT TRANSFORM ESTIMATES

The variance in the estimate $\hat{R}_{xy}(\tau_o)$ associated with Equations (125) or (130) is given approximately by

$$\sigma^2[\hat{R}_{xy}(\tau_o)] \approx \frac{1}{Nn_d} [R_{xx}(0) R_{yy}(0) + R_{xy}^2(\tau_o)] \quad (131)$$

A similar formula gives the variance for $\hat{R}_{x\tilde{y}}(\tau_o)$ where $\tilde{y}(t) = \mathcal{H}[y(t)]$ replaces $y(t)$, namely,

$$\sigma^2[\hat{R}_{x\tilde{y}}(\tau_o)] \approx \frac{1}{Nn_d} [R_{xx}(0) R_{\tilde{y}\tilde{y}}(0) + R_{x\tilde{y}}^2(\tau_o)] \quad (132)$$

But, from relations in Table 3, it is known that

$$R_{x\tilde{y}}(\tau) = \tilde{R}_{xy}(\tau) \text{ and } R_{\tilde{y}\tilde{y}}(0) = R_{yy}(0) \quad (133)$$

Hence

$$\sigma^2[\hat{\tilde{R}}_{xy}(\tau_o)] \approx \frac{1}{Nn_d} [R_{xx}(0) R_{yy}(0) + \tilde{R}_{xy}^2(\tau_o)] \quad (134)$$

However, at the maximum value, $\tilde{R}_{xy}(\tau_o) = 0$ so that

$$\sigma^2[\hat{\tilde{R}}_{xy}(\tau_o)] \approx \frac{1}{Nn_d} [R_{xx}(0) R_{yy}(0)] \quad (135)$$

Equations (131) and (135) show that under all situations, the variance in estimates of $\hat{\tilde{R}}_{xy}(\tau_o)$ are bounded by the variance in estimates of $R_{xy}(\tau_o)$.

$$\sigma^2[\hat{\tilde{R}}_{xy}(\tau_o)] \leq \sigma^2[\hat{R}_{xy}(\tau_o)] \quad (136)$$

To explore this matter further, let

$\sigma_1 = \sigma[\hat{R}_{xy}(\tau_o)]$ = standard deviation in estimates $\hat{R}_{xy}(\tau_o)$ of the true peak value $R_{xy}(\tau_o)$

$\sigma_2 = \sigma[\hat{\tilde{R}}_{xy}(\tau_o)]$ = standard deviation in estimates $\hat{\tilde{R}}_{xy}(\tau_o)$ of the true zero value $\tilde{R}_{xy}(\tau_o)$

Assuming Equations (131) and (135) to apply, then

$$\sigma_1 \approx \left[\frac{\sigma_x^2 \sigma_y^2 + R_{xy}^2(\tau_o)}{Nn_d} \right]^{1/2} = R_{xy}(\tau_o) \left[\frac{1 + \rho_{xy}^{-2}(\tau_o)}{Nn_d} \right]^{1/2} \quad (137)$$

$$\sigma_2 \approx \left[\frac{\sigma_x^2 \sigma_y^2}{Nn_d} \right]^{1/2} = R_{xy}(\tau_o) \left[\frac{\rho_{xy}^{-2}(\tau_o)}{Nn_d} \right]^{1/2} \quad (138)$$

where $\rho_{xy}(\tau_o)$ is defined, as in Equation (103), by

$$\rho_{xy}(\tau_o) = \frac{R_{xy}(\tau_o)}{\sqrt{R_{xx}(0) R_{yy}(0)}} = \frac{R_{xy}(\tau_o)}{\sigma_x \sigma_y} \quad (139)$$

Consider the same illustration values used earlier of $\rho_{xy}(\tau_o) = 0.60$ and $Nn_d = 9,444$. For these parameters, Equations (137) and (138) give $\sigma_1 \approx 0.02 R_{xy}(\tau_o)$ and $\sigma_2 \approx 0.017 R_{xy}(\tau_o)$ with the following interpretations. There will be approximately 95% confidence from a single estimate $\hat{R}_{xy}(\tau_o)$ that the true peak value $R_{xy}(\tau_o)$ lies in the interval

$$[0.96 \hat{R}_{xy}(\tau_o) \leq R_{xy}(\tau_o) \leq 1.04 \hat{R}_{xy}(\tau_o)]$$

There will be approximately 95% confidence from a single estimate $\hat{R}_{xy}(\tau_o)$ that the true zero value $\tilde{R}_{xy}(\tau_o)$ lies in the interval

$$[-0.034 \hat{R}_{xy}(\tau_o) \leq \tilde{R}_{xy}(\tau_o) \leq 0.034 \hat{R}_{xy}(\tau_o)]$$

6.3. ENVELOPE OF CORRELATION FUNCTION ESTIMATES

Consider next the envelope function $A_{xy}(\tau)$ computed from

$$A_{xy}(\tau) = [R_{xy}^2(\tau) + \tilde{R}_{xy}^2(\tau)]^{1/2} \quad (140)$$

At the maximum point $\tau = \tau_o$, $\tilde{R}_{xy}(\tau_o) = 0$ and

$$A_{xy}(\tau_o) = R_{xy}(\tau_o) \quad (141)$$

It follows that the normalized random error for estimates of $A_{xy}(\tau_o)$ and $R_{xy}(\tau_o)$ will be equal. Thus

$$\epsilon [\hat{A}_{xy}(\tau_o)] \approx \frac{[1 + \rho_{xy}^{-2}(\tau_o)]^{1/2}}{(Nn_d)^{1/2}} \quad (143)$$

Also for small ϵ , with 95% confidence, one can state that the true maximum value $A_{xy}(\tau_o)$ lies in the interval

$$[(1 - 2\epsilon) \hat{A}_{xy}(\tau_o) \leq A_{xy}(\tau_o) \leq (1 + 2\epsilon) \hat{A}_{xy}(\tau_o)] \quad (144)$$

Table 4 applies both to estimates $\hat{R}_{xy}(\tau_o)$ and $\hat{A}_{xy}(\tau_o)$.

7. STATISTICAL ERRORS IN LOCATION ESTIMATES

Results in Section 6 give normalized random error formulas that indicate how well one can estimate the magnitudes of peak values $R_{xy}(\tau_o)$ and $A_{xy}(\tau_o)$, and the variance in estimates of zero values of the Hilbert transform $\hat{R}_{xy}(\tau_o)$. This section will now discuss the related problem of determining the location $\tau = \tau_o$ where the peak values of $R_{xy}(\tau)$ and $A_{xy}(\tau)$ occur, as well as where the zero crossings of $\hat{R}_{xy}(\tau)$ occur.

For definiteness, near the maximum point $\tau = \tau_o$, it will be assumed that $R_{xy}(\tau)$ has the form associated with band-limited white noise as per Equation (112), namely,

$$R_{xy}(\tau) = R_{xy}(\tau_o) \left[\frac{\sin 2\pi B(\tau - \tau_o)}{2\pi B(\tau - \tau_o)} \right] \quad (145)$$

To simplify notation and analysis, without loss of generality, let $\tau_o = 0$ so that Equation (145) becomes

$$R_{xy}(\tau) = R_{xy}(0) \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right) \quad (146)$$

The problem at issue is to find the location $\tau = 0$ where $R_{xy}(\tau)$ of Equation (146) achieves its maximum value. This answer will be the same as finding the location $\tau = \tau_o$ where the $R_{xy}(\tau)$ of Equation (145) achieves its maximum value.

7.1. CORRELATION FUNCTION ESTIMATES

Near $\tau = 0$, expansion of $\sin 2\pi B\tau$ shows

$$\sin 2\pi B\tau \approx (2\pi B\tau) - \frac{(2\pi B\tau)^3}{6} \quad (147)$$

Hence, near $\tau = 0$, Equation (146) gives estimates

$$\hat{R}_{xy}(\tau) \approx \left[1 - \frac{2}{3} (\pi B\tau)^2 \right] R_{xy}(0) \quad (148)$$

with mean value

$$E[\hat{R}_{xy}(0)] = R_{xy}(0) \quad (149)$$

and variance

$$\sigma^2[\hat{R}_{xy}(0)] \approx \frac{4}{9} (\pi B)^4 R_{xy}^2(0) E[\tau^4] \quad (150)$$

Assume next that values of τ near $\tau = 0$ follow a Gaussian distribution with zero mean value $m(0)$ and variance $\sigma_1^2(0)$, namely,

$$m(0) = E[\tau] = 0 \quad \sigma_1^2(0) = E[\tau^2] \quad (151)$$

For such Gaussian data, the fourth moment

$$E[\tau^4] = 3\{E[\tau^2]\}^2 = 3\sigma_1^4(0) \quad (152)$$

Substitution into Equation (150) yields

$$\sigma^2[\hat{R}_{xy}(0)] \approx \frac{4}{3}(\pi B)^4 R_{xy}^2(0) \sigma_1^4(0) \quad (153)$$

Hence, the standard deviation

$$\sigma_1(0) \approx \frac{0.93}{(\pi B)} \{ \epsilon[\hat{R}_{xy}(0)] \}^{1/2} \quad (154)$$

when the maximum point is at $\tau_o = 0$. For an arbitrary maximum point $\tau = \tau_o$, Equation (154) represents values of τ near $\tau = \tau_o$ and is replaced by

$$\sigma_1(\tau_o) \approx \frac{0.93}{(\pi B)} \{ \epsilon[\hat{R}_{xy}(\tau_o)] \}^{1/2} \quad (155)$$

where $\epsilon[\hat{R}_{xy}(\tau_o)]$ is computed from Equation (130). Thus, Equation (154) is a special case of Equation (155). The 95% confidence interval for determining the location where the peak value occurs is given by

$$[\tau_o - 2\sigma_1(\tau_o) \leq \tau \leq \tau_o + 2\sigma_1(\tau_o)] \quad (156)$$

using $\sigma_1(\tau_o)$ of Equation (155).

To illustrate these results, suppose $\epsilon[\hat{R}_{xy}(\tau_o)] = 0.10$ and $B = 10$ Hz. Then Equation (155) yields $\sigma_1(\tau_o) \approx 0.0094$ seconds. The 95% confidence interval here for the location of the peak value will be the interval from $(\tau_o - 0.019)$ seconds to $(\tau_o + 0.019)$ seconds. As another example, given the value of B , one can solve for the required $\epsilon[\hat{R}_{xy}(\tau_o)]$ to achieve a desired value of $\sigma_1(\tau_o)$.

7.2. HILBERT TRANSFORM ESTIMATES

The Hilbert transform corresponding to $R_{xy}(\tau)$ of Equation (146) is given by

$$\tilde{R}_{xy}(\tau) = R_{xy}(0) \left[\frac{1 - \cos 2\pi B\tau}{2\pi B\tau} \right] \quad (157)$$

Near $\tau = 0$, expansion of $\cos 2\pi B\tau$ shows

$$\cos 2\pi B\tau \approx 1 - \frac{(2\pi B\tau)^2}{2} \quad (158)$$

Hence, near $\tau = 0$, Equation (157) gives estimates

$$\hat{R}_{xy}(\tau) \approx (\pi B\tau) R_{xy}(0) \quad (159)$$

with zero mean value and variance

$$\sigma^2[\hat{R}_{xy}(0)] \approx (\pi B)^2 R_{xy}^2(0) E[\tau^2] \quad (160)$$

where $E[\tau^2]$ may not be the same as the previous result in Equation (151).

Assume here that values of τ near $\tau = 0$ follow a Gaussian distribution with zero mean value and variance $\sigma_2^2(0) = E[\tau^2]$ where $\sigma_2^2(0)$ is used instead of the previous $\sigma_1^2(0)$ of Equation (151) to distinguish this variance in locating zero crossings of $\hat{R}_{xy}(0)$ from the previous variance in locating maximum values of $R_{xy}(0)$. For an arbitrary zero crossing point $\tau = \tau_o$, Equation (160) represents values of τ near $\tau = \tau_o$ and is replaced by

$$\sigma^2[\hat{R}_{xy}(\tau_o)] \approx (\pi B)^2 R_{xy}^2(\tau_o) \sigma_2^2(\tau_o) \quad (161)$$

From Equation (136), it follows that

$$(\pi B)^2 R_{xy}^2(\tau_o) \sigma_2^2(\tau_o) \leq \sigma^2[\hat{R}_{xy}(\tau_o)] \quad (162)$$

This yields

$$\sigma_2(\tau_o) \leq \frac{1}{(\pi B)} \epsilon[\hat{R}_{xy}(\tau_o)] \quad (163)$$

with the 95% confidence interval

$$[\tau_o - 2\sigma_2(\tau_o) \leq \tau \leq \tau_o + 2\sigma_2(\tau_o)] \quad (164)$$

Comparison of Equations (155) and (163) now shows that $\sigma_2(\tau_o)$ can be approximated by

$$\sigma_2(\tau_o) \approx \sigma_1(\tau_o) \{ \epsilon[\hat{R}_{xy}(\tau_o)] \}^{1/2} \quad (165)$$

In particular, $\sigma_2(\tau_o)$ will be less than $\sigma_1(\tau_o)$ whenever $\epsilon[\hat{R}_{xy}(\tau_o)]$ is less than unity. For example, if $\epsilon[\hat{R}_{xy}(\tau_o)] \approx 0.10$, then $\sigma_2(\tau_o) \approx 0.316 \sigma_1(\tau_o)$. Suppose also that $B = 10$ Hz. Then, as noted earlier, Equation (155) gives $\sigma_1(\tau_o) \approx 0.0094$ seconds. However, $\sigma_2(\tau_o)$ will be approximately 0.003 seconds leading to a narrower 95% confidence interval for the location of the point where the zero value of $\hat{R}_{xy}(\tau_o)$ occurs than for the location of the point where the peak value of $R_{xy}(\tau_o)$ occurs.

7.3. ENVELOPE OF CORRELATION FUNCTION ESTIMATES

The envelope function corresponding to $R_{xy}(\tau)$ of Equation (146) is given by

$$A_{xy}(\tau) = A_{xy}(0) \left| \frac{\sin \pi B \tau}{\pi B \tau} \right| \quad (166)$$

using the fact that $A_{xy}(0) = R_{xy}(0)$. Expansion of $\sin \pi B \tau$ near $\tau = 0$ gives the estimate

$$\hat{A}_{xy}(\tau) \approx \left[1 - \frac{(\pi B \tau)^2}{6} \right] A_{xy}(0) \quad (167)$$

with mean value

$$E[\hat{A}_{xy}(0)] = A_{xy}(0) \quad (168)$$

and variance

$$\sigma^2[\hat{A}_{xy}(0)] \approx \frac{(\pi B)^4}{36} A_{xy}^2(0) E[\tau^4] \quad (169)$$

Assume here that values of τ near $\tau = 0$ follow a Gaussian distribution with zero mean value and variance $\sigma_3^2(0) = E[\tau^2]$ which may be different than the previous $\sigma_1^2(0)$ or $\sigma_2^2(0)$. Now

$$E[\tau^4] = 3\sigma_3^4(0) \quad (170)$$

Substitution into Equation (169) yields

$$\sigma^2[\hat{A}_{xy}(0)] \approx \frac{(\pi B)^4}{12} A_{xy}^2(0) \sigma_3^4(0) \quad (171)$$

Hence, the standard deviation

$$\sigma_3(0) \approx \frac{1.86}{(\pi B)} \{ \epsilon[\hat{A}_{xy}(0)] \}^{1/2} \quad (172)$$

when the maximum point is at $\tau = 0$. For an arbitrary point $\tau = \tau_o$, Equation (172) represents values of τ near $\tau = \tau_o$ and is replaced by

$$\sigma_3(\tau_o) \approx \frac{1.86}{(\pi B)} \{ \epsilon[\hat{A}_{xy}(\tau_o)] \}^{1/2} \quad (173)$$

As noted earlier in Equation (142), one also has $\epsilon[\hat{A}_{xy}(\tau_o)] = \epsilon[\hat{R}_{xy}(\tau_o)]$. Thus, comparison of Equations (155) and (173) yields

$$\sigma_3(\tau_o) \approx 2\sigma_1(\tau_o) \quad (174)$$

In words, the standard deviation in locating where peak values of $A_{xy}(\tau)$ occur is approximately twice the standard deviation in locating where peak values of $R_{xy}(\tau)$ occur.

8. DISPERSIVE PROPAGATION PROBLEMS

Previous material in Sections 5 to 7 applies to nondispersive propagation problems where the velocity of propagation is a constant independent of frequency. Consider other situations where the propagation paths are frequency dispersive as discussed in References [3, 18, 19]. In particular for flexural waves in structures, the “apparent” propagation speed of the waves at a given frequency is called the **group velocity** c_g . This c_g is related but not equal to the **phase velocity** c_p . It is known that the group velocity of flexural waves in thin beams satisfies

$$c_g = 2c_p \sim \sqrt{f} \quad (175)$$

In words, c_g is twice c_p and both are proportional to the square root of frequency.

Nondispersive Formulas

To show how dispersive results can differ from nondispersive results, return to the basic nondispersive propagation model in Section 5.2. Let $x(t)$ be narrow-band white noise as specified in Equations (113) and (114). For simplicity, assume that both $m(t)$ and $n(t)$ are negligible in Equations (97) and (98) so that

$$\begin{aligned} x(t) &= s(t) \\ y(t) &= \alpha s(t - \tau_o) \end{aligned} \quad (176)$$

Here

$$R_{xx}(\tau) = R_{ss}(\tau) = KB_o \left(\frac{\sin \pi B_o \tau}{\pi B_o \tau} \right) \cos 2\pi f_o \tau \quad (177)$$

$$R_{yy}(\tau) = \alpha^2 R_{ss}(\tau) \quad (178)$$

$$R_{xy}(\tau) = \alpha R_{ss}(\tau - \tau_o) \quad (179)$$

Hence, the cross-correlation coefficient function of Equation (103) becomes

$$\rho_{xy}(\tau) = \left[\frac{\sin \pi B_o (\tau - \tau_o)}{\pi B_o (\tau - \tau_o)} \right] \cos 2\pi f_o (\tau - \tau_o) \quad (180)$$

where

$$\tau_o = (d/c) = \text{constant} \quad (181)$$

Equation (180) is plotted in Figure 7.

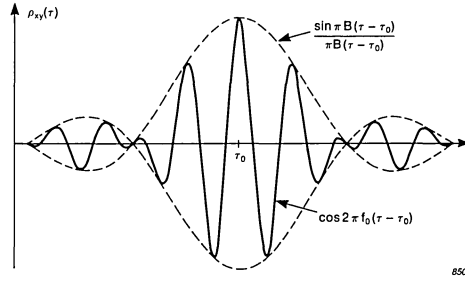


Fig. 7. Cross-Correlation for Nondispersive Propagation

A number of properties of Equation (180) for nondispersive propagation are worthy of note.

1. The cosine function $\cos 2\pi f_0 \tau$ at the center frequency f_0 of the band $[f_0 - (B_0/2)]$ to $[f_0 + (B_0/2)]$ is modulated by the envelope function $(\sin \pi B_0 \tau) / \pi B_0 \tau$ that is defined by the bandwidth B_0 .
2. The peak value of the envelope function occurs at the time delay $\tau_0 = (d/c)$, a fixed value independent of frequency.
3. Peak values of the cosine function occur at time delays $\tau_n = \tau_0 \pm (n/f_0)$ where n is any integer. In particular, a peak value of the cosine function coincides with the peak value of the envelope function.
4. The main lobe of the envelope function extends from $[\tau_0 - (1/B_0)]$ to $[\tau_0 + (1/B_0)]$, a width of $(2/B_0)$ Hz. The number of oscillations of the cosine function contained in the main lobe of the envelope function is $(2 f_0 / B_0)$.

Dispersive Formulas

For dispersive propagation problems governed by Equation (175), as a first order of approximation, Reference [19] proves that the cross-correlation coefficient function corresponding to Equation (180) now takes the form

$$\rho_{xy}(\tau) \approx \left[\frac{\sin \pi B_0 (\tau - \tau_2)}{\pi B_0 (\tau - \tau_2)} \right] \cos 2\pi f_0 (\tau - \tau_1) \quad (182)$$

where

$$\tau_1 = (d/c_p) \sim f_0^{-1/2} \quad (183)$$

$$\tau_2 = (d/c_g) \sim f_0^{-1/2} \quad (184)$$

with $\tau_1 = 2\tau_2$ since $c_g = 2c_p$.

Equation (182) is plotted in Figure 8.

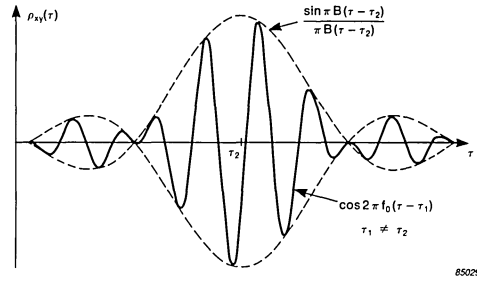


Fig. 8. Cross-Correlation for Dispersive Propagation

Observe that Equation (182) is similar in nature to Equation (180) but has **two** important differences.

1. The peak value of the envelope function occurs at the time delay $\tau_2 = (d/c_g)$ which is now a function of frequency since $c_g \sim \sqrt{f_o}$.
2. Peak values of the cosine function occur at time delays $\tau_n = \tau_1 \pm (n/f_o)$ where n is any integer. In general, a peak value of the cosine function does **not** coincide with the peak value of the envelope function.

Equation (182) shows that the “apparent” propagation speed for the envelope function at a given frequency is determined by the group velocity c_g rather than by the phase velocity c_p . For dispersive propagation problems, as illustrated by Equation (182), the peak value due to fine details in a cosine function is usually **not** at the same location as the desired envelope peak value. To find where the envelope peak value occurs, one should employ the Hilbert transform techniques discussed in this paper.

This concludes the paper.

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